Effects of time-delayed interactions on dynamic patterns in a coupled phase oscillator system

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(Received 16 February 1999)

We investigate the effects of time delayed interactions in the network of neural oscillators. We perform the stability analysis in the vicinity of a synchronized state at vanishing time delay and present a related phase diagram. In the simulations it is shown that time delay induces various phenomena such as clustering where the system is spontaneously split into two phase locked groups, synchronization, and multistability. Time delay effects should be considered both in the natural and artificial neural systems whose information processing is based on the spatiotemporal dynamic patterns. [S1063-651X(99)07309-2]

PACS number(s): 05.45.Xt, 87.18.Sn

I. INTRODUCTION

When information is processed in natural neural systems, there is a time delay corresponding to the conduction time of action potentials along the axons. With this physiological background the time delay effects have been theoretically investigated in several neural oscillator models. In Ref. [1] a delay has been introduced to investigate the phase dynamics of oscillators in a two-dimensional layer in the context of temporal coding. The two neuron model with a time delay has been analytically studied by Schuster *et al.* [2] focusing on the entrainment of the oscillators due to delay. A delay has been shown to influence the existence and the stability of metastable states in two-dimensional oscillators with nearest-neighbor coupling [3]. Recently, it has been shown that the time delay induces the multistability in coupled oscillator systems which may provide a possible mechanism for the perception of ambiguous or reversible figures [4].

In this paper we analytically and numerically investigate the time delay effects on dynamic patterns in globally coupled oscillators. Dynamic patterns such as phase locking and clustering represent the collective properties of neurons participating in the information processing. To investigate the time delay effects on these patterns occurring in the neural systems we choose a phase oscillator model. In particular, the phase interactions with more than a first Fourier mode will be considered to describe the rich dynamic patterns. The significance of higher mode phase interactions may be found in the nontrivial dynamics in coupled neurons [5]. The phase model with first and second Fourier interaction modes which will be considered in this paper has been introduced to understand the pattern-forming dynamics in brain behavior [6]. The existence and the stability of clustering state of coupled neural oscillators have been studied in the same model [7].

In Sec. II, we perform the stability analysis of synchronized state. The analytical results do not depend on the number of oscillators. We present a phase diagram of parameters indicating the stable and unstable regions of the wholly synchronized state. Numerical results are presented in Sec. III. Various phenomena such as clustering, synchronization, and multistability between the synchronized state and clustered state induced by time delay are presented. Section IV is devoted to a discussion with summarized results.

II. STABILITY ANALYSIS OF SYNCHRONIZED STATE

We consider the overdamped oscillator model with first and second Fourier interaction which is given by the following equation of motion [7]:

$$\frac{d\phi_i}{dt}=\omega+\frac{g}{N}\sum_{j=1}^{N}\{-\sin[\phi_i(t)-\phi_j(t-\tau)+a] + r\sin 2[\phi_i(t)-\phi_j(t-\tau)]\}, \tag{1}$$

ϕ_i $(i=1,2,...,N,\ 0\leqslant\phi_i<2\pi)$ is the phase of the ith oscillator, ω is the uniform intrinsic frequency of the oscillators, g/N is the global coupling of the oscillators scaled down by the number of the oscillators, and τ is the time delay. The interaction of the system is characterized by parameters a and r as well. With the symmetry of the system we safely take a in the range $[0, \pi]$. We also consider $g>0$ (the excitatory coupling) in this paper.

The first Fourier mode in Eq. (1) without time delay is an attractive interaction which yields the synchronization of the system, while the second tends to desynchronize the system. The competition between these two interactions generates nontrivial dynamic patterns. Without time delay the synchronized system bifurcates into two cluster state at critical parameter values. For any coupling constant $2r=\cos a$ defines the critical line where the instability of the phase locked state occurs.

We assume a synchronized state $\phi_i(t)=\Phi(t)=\Omega t$. Then Eq. (1) gives

$$\Omega=\omega+g[-\sin(\Omega\tau+a)+r\sin(2\Omega\tau)]. \tag{2}$$

To analyze the stability of this synchronized state, we deviate ϕ_i and linearize the system around the synchronized state. That is, for $\phi_i(t)=\Omega t+\delta\phi_i(t)$,

$$\frac{d(\delta\phi_i)}{dt}=\frac{g}{N}[2r\cos(2\Omega\tau)-\cos(\Omega\tau+a)]\times\sum_{j=1}^{N}[\delta\phi_i(t)-\delta\phi_j(t-\tau)]. \tag{3}$$

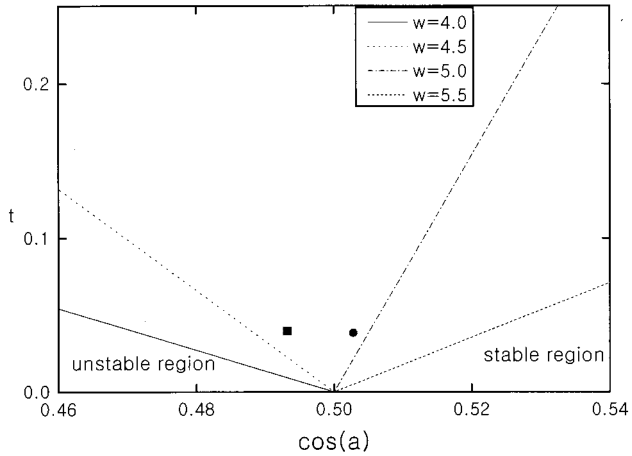

FIG. 1. Plot of critical lines defining the stability of synchronized state when $g=5.6$ and $r=0.25$. It can be seen that at some nonzero time delay values the synchronized state for $\tau=0$ can be unstable when $2r<\cos a$ and the dephased state for $\tau=0$ can be synchronized when $2r>\cos a$. The corresponding examples are denoted as the solid circle and square, respectively.

Converting the above equation to the eigenvalue problem, one obtains

$$\lambda a_i=\frac{g}{N}[2r\cos(2\Omega\tau)-\cos(\Omega\tau+a)]\sum_{j=1}^{N}[a_i-a_j \times\exp(-\lambda\tau)], \quad (4)$$

where $\delta\phi_i(t)=a_i\exp(\lambda t)$. If $\Sigma_{i=1}^{N}a_i\neq 0$, one obtains for $\lambda<0$

$$2r>\cos a \quad (5)$$

in the $\tau\to 0$ limit. This contradicts to the stability condition when $\tau=0$. Therefore, $\Sigma_{i=1}^{N}a_i=0$, and

$$\lambda=g[2r\cos(2\Omega\tau)-\cos(\Omega\tau+a)]. \quad (6)$$

To visualize the time delay effects on the stability of the synchronized state, we take a small τ. In the $\tau\to 0$ limit, Eqs. (2) and (6) give

$$\lambda=g[2r-\cos a+\sin a(\omega-g\sin a)\tau]+O(\tau^2). \quad (7)$$

Therefore, for $\omega>g\sqrt{1-4r^2}$, $\lambda=0$ in Eq. (6), the critical line separating the stable region of the synchronized state from the unstable one, is in the parameter range of

$$2r<\cos a. \quad (8)$$

However, for strong coupling, $\omega<g\sqrt{(1-4r^2)}$, the critical line lies in the realm of

$$2r>\cos a. \quad (9)$$

Therefore, for strong coupling, the stability condition is drastically changed even when τ is very small. In Fig. 1, we plot the phase diagram of Eq. (7) in the parameter space of $\cos a$, and τ for several values of ω with fixed coupling constant. $\cos a=0.5$ in Fig. 1 is the critical line when $\tau=0$.

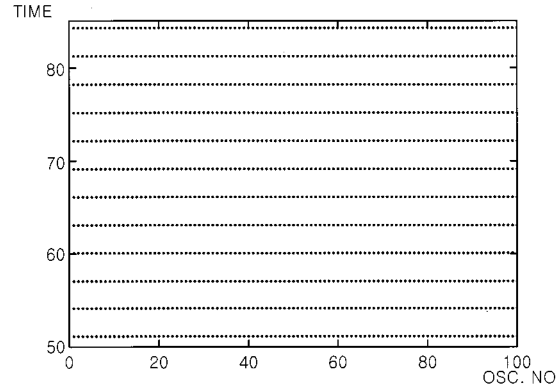

FIG. 2. Times evolution of oscillator phases when $\omega=5.0$, $g=5.6$, $a=1.04$, $r=0.25$, and $\tau=0$. The dots represent 0 phase timing of each oscillator. The system is at the synchronized state.

III. NUMERICAL RESULTS

In the last section we performed the stability analysis around the synchronized ansatz, $\phi_i(t)=\phi(t)=\Omega t$, whose results are independent of N, the number of the oscillators. In this section we investigate numerically the time delay effects on the dynamical patterns of the system. To this end, we choose parameter values where the system is realized at a synchronized state for vanishing time delay. For fixed parameter values the system exhibits various dynamical patterns as time delay values change. We also study the time delay effects at parameter values where the system is desynchronized at zero time delay. In the simulations, we have used the fourth-order Runge-Kutta method with discrete time step of $\Delta t=0.005$ with random initial conditions.

We consider the system with the parameter values given by $\omega=5.0$, $g=5.6$, $a=1.04$, and $r=0.25$, where the synchronized ansatz is stable for vanishing time delay value. In Fig. 2 we plot the time evolution of the system at $\tau=0$ for $N=100$. The system is always realized as the synchronized state when there is no time delay.

At $\tau=0.15$ the system is at clustered state, whose time evolution is shown in Fig. 3. The average ratio of the oscillator population in the two groups is 1:1. Clustering in this paper is induced by time delay which results in the inhibitory coupling effects. The phase difference between the two separated groups of oscillators depends on the time delay values. In Fig. 4 we plot the order parameter defined by

$$O(t)=\frac{2}{N(N-1)}\sum_{i,j}^{N}\sin\left(\frac{|\phi_i-\phi_j|}{2}\right) \quad (10)$$

for three different values of τ where the system is at clustered state.

At $\tau=0.5$ the system may dwell on a wholly synchronized state or clustered state according to the initial conditions. This is a manifestation of multistability. We plot $O(t)$ in Fig. 5 showing the multistability. The dotted line in Fig. 5 repre-

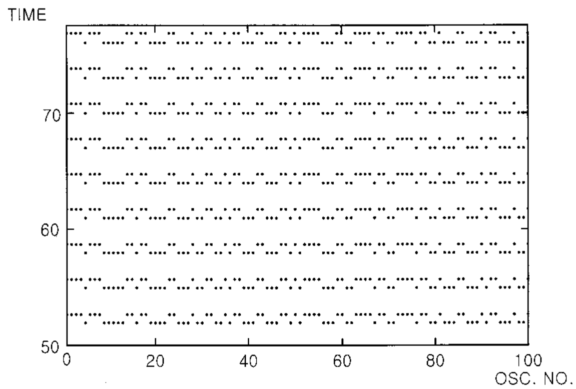

FIG. 3. Time evolution of oscillator phases when $\tau=0.15$ with the same parameter values as in Fig. 2. The average ratio of the oscillator populations in the two clusters is 1:1.

sent the clustering state where the clusters are not uniformly moving. Therefore, the phase difference between the two clusters is not fixed but oscillatorically changed. In Fig. 6 we plot the evolution of ϕ_i's corresponding to the dotted line in Fig. 5. The motion of the clusters is not uniform but the velocity of the clusters depends on the phase value of the clusters. While in Ref. [4] the multistability exists either between the synchronized state and a desynchronized state where the oscillators are distributed almost uniformly between the synchronized states with different moving frequencies, the multistability in this paper is realized between the synchronized state and the clustered state. For $0.6<\tau<2.5$, the system exists always in the perfectly synchronized state.

We study the time delay effects on the desynchronized state when there is no time delay. We choose $a=1.25$ with the other parameter values same as above. When $\tau=0.49$, the system shows multistability between the synchronized state and the clustered state. For $0.5<\tau<2.5$ oscillators are perfectly synchronized. This is a synchronization induced by time delay.

IV. SUMMARY AND DISCUSSIONS

In this paper we investigated the time delay effects on the dynamic patterns in the coupled phase oscillators with first and second Fourier mode interactions. The analytical study shows that the time delay drastically changes the stability of the synchronized state. To investigate the time delay effects on dynamical patterns numerically, we fixed all the other parameters than the time delay. The introduction of time delay into a fully synchronized state induces the clustering, and the multistability. Time delay also induces synchronization of desynchronized state when there is no time delay. This shows that the time delay may play an important role in the information processing based on the spatiotemporal structure of neuronal activities [8]. The results in this paper suggest that the time delay is a route leading to such dynamic patterns. It is expected that the time delay may provide a rich structure of dynamics which may be used to facilitate memory retrieval when the informations are stored on the basis of dynamics of the system [9].

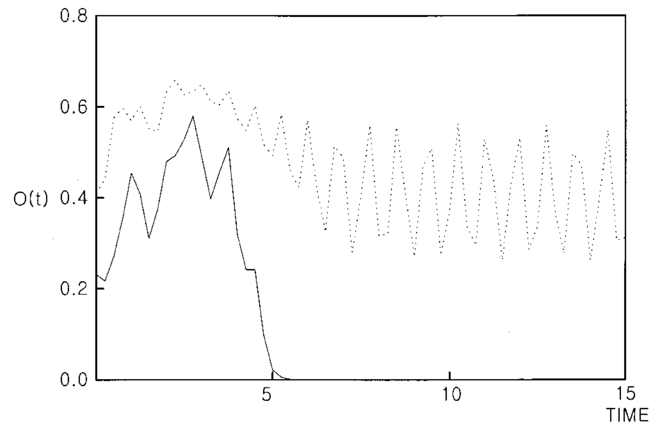

FIG. 5. Plot of the order parameter $O(t)$ for $\tau=0.5$ showing the multistability of the system. The dotted line represents the clustered state where the distance between the two clusters is changing because of the nonuniform motion of the clusters.

ACKNOWLEDGMENTS

This work was supported by the Ministry of Information and Communications, Korea.

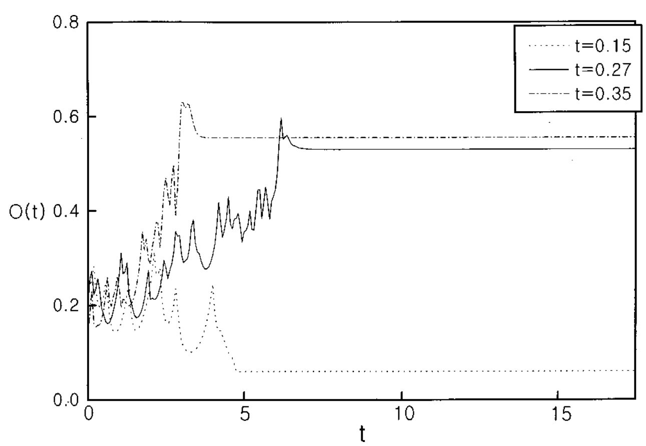

FIG. 4. Plot of the order parameter $O(t)$ for three values of time delay. $O(t)$ represents the distance of the two clusters. The graphs show that the clusters are uniformly moving at the steady states so that $O(t)$ is constant.

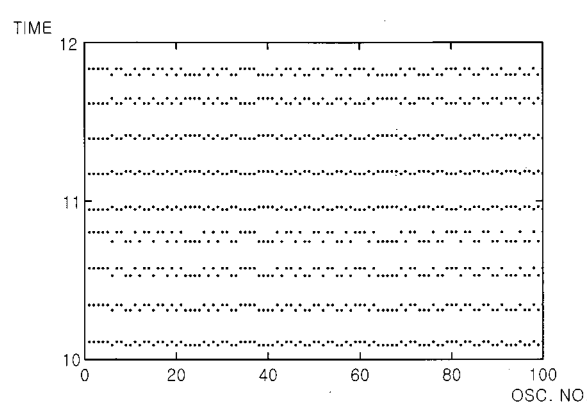

FIG. 6. Time evolution of oscillator phases corresponding to the dotted line in Fig. 5. The distance between the two clusters changes periodically.

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